BRANCHING BROWNIAN MOTIONS

SHEN JINGYUN

(Bachelor of Science, Shanghai Jiao Tong University)

A THESIS SUBMITTED

FOR THE DEGREE OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

NATIONAL UNIVERSITY OF SINGAPORE

2012

DECLARATION

I hereby declare that the thesis is my original work and it has been written by me in its entirety.

I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

Shen Jingyun

13 August 2012

Acknowledgements

First and foremost, I would like to show my deepest gratitude to my supervisor, Dr. Sun Rongfeng, a respectable, responsible and resourceful scholar, who has provided me with valuable guidance in every stage of the writing of this thesis. Without his enlightening instruction, impressive kindness and patience, I could not have completed my thesis. His keen and vigorous academic observation enlightens me not only in this thesis but also in my future study.

I shall extend my thanks to all my teachers who have helped me to develop the fundamental and essential academic competence.

Last but not least, I'd like to thank my families, my friends, and my boyfriend, for their constant encouragement and support.

Table of Contents

Summary

Branching Brownian Motion (BBM) describes a system of particles where each particle moves according to a Brownian motion and has an exponentially distributed lifetime.

Branching Brownian Motion has a long history. In 1937, Kolmogorov, Petrovsky, and Piscounov proved the first order term of the Maximal Displacement is $\sqrt{2}t + o(t)$ as $t \to \infty$.

In 1977, Bramson proved the second-order term is $\sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + O(1)$ as $t \to \infty$. He then developed an important relation between branching Brownian motions and the Fisher-KPP equation based on the work of McKean.

Roberts provided another proof of Bramson's result concerning the median position of the extremal particle in branching Brownian motions, which is much shorter than Bramson's proof.

L.-P Arguin, A. Bovier and N. Kistler work on the extremal process of branching Brownian motion. They addressed the genealogy of extremal particles. They proved that in the limit of large time t, extremal particles descend with overwhelming probability from ancestors having split either within a distance of order one from time 0, or within a distance of order one from time t. The results suggest that the extremal process of branching Brownian motion in the limit of large times converges weakly to a cluster point process. The limiting process is a randomly shifted Poisson cluster process.

The work of E.Aidekon, J. Berestycki, E. Brunet and Z. Shi also gives a description of the limit object and a different proof of the convergence of the branching Brownian motion seen from its tip.

This thesis shows the main results on the branching Brownian motion till now. We describe how these results are obtained and mainly focus on the proof strategies.

List of Tables

List of Figures

Chapter 1 Introduction

1.1 The problem

Branching Brownian motion has a long history. A general theory of branching Markov processes was developed in a series of three papers by Ikeda, Nagasawa and Watanabe in 1968, 1969 [8,9,10].

Branching Brownian Motion (BBM) describes a system of particles where each particle moves according to a Brownian motion and has an exponentially distributed lifetime. It is a continuous time Markov branching process with death rate 1, producing a random number of offsprings at its time of death, where the individual particles undergo independent standard Brownian motions. It is constructed as follows: a single particle performs a standard Brownian motion x, with x(0)=0, which it continues for an exponential holding time T independent of x, with $\mathbb{P}[T > t] = e^{-t}$. At time T, the particle splits independently of x and T into k (≥1) offsprings with probability p_k, where $\sum_{k=1}^{\infty} p_k = 1$, $\sum_{k=1}^{\infty} kp_k = 2$, and $K = \sum_{k=1}^{\infty} k(k-1)p_k < \infty$. These particles continue along independent Brownian paths starting at X(T), and are subject to the same splitting rule, with the effect that the resulting tree X contains, after an elapsed time $t > 0$, N(t) particles located at $X_1(t), \dots, X_{N(t)}(t)$, where N(t) is the random number of particles generated up to that time (note that $\mathbb{E}[N(t)] = e^t$). And denote $\mathcal{N}(t) = \{X_k(t), 1 \leq k \leq N(t)\}$ the collection of positions of particles in branching Brownian motion at time t.

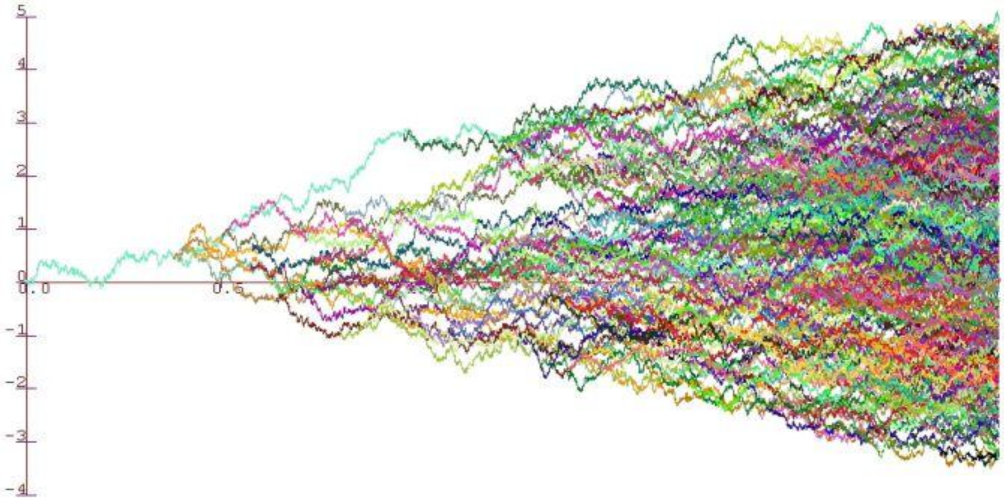

Fig 1.1 A branching Brownian motion in one dimension [14]

Define $M_t = \max_{1 \le k \le N(t)} X_k(t)$, and denote

$$u(t, x) = \mathbb{P}[M_t \le x],$$

the distribution function of the maximal displacement of the branching Brownian motions at time t.

For $0 < \varepsilon < 1$, let $m_\varepsilon(t)$ be chosen to satisfy $u\big(t, m_\varepsilon(t)\big) = \mathbb{P}[M_t \le m_\varepsilon(t)] = \varepsilon$. The asymptotic behavior of $u(t, x)$ will be studied.

1.2 The objective

Much current research in probability theory is concerned with branching processes. These generalizations are essential for the modeling of systems. Branching Brownian motion is most easily described as a simple model of an evolving population.

A number of different ways of thinking about branching Brownian motions have emerged over the last ten years, and the principal aim of this thesis is to describe them in an accessible way. We will mainly focus on the proof strategies and key points while the details of most proofs might be omitted, and the reader is referred to the relevant papers.

Chapter 2 Bramson's main results

In Bramson's Ph.D. dissertation in 1977 [5], he shows that the position of any fixed percentile of the maximal displacement of standard branching Brownian motion in one dimension is $\sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + O(1)$ as $t \to \infty$, where the second-order term was previously unknown. He then developed an important relation between branching Brownian motions and the Fisher-KPP equation based on the work of McKean [13].

2.1 Results of Kolmogorov, Petrovsky, and Piscounov

Theorem 2.1 [5, page 579-580]: u(t,x) solves the Kolmogorov-Petrovsky-Piscounov (KPP) equation

$$u_t = \frac{1}{2}u_{xx} + f(u),$$

with initial condition

$$(2.1) \quad u(0,x) = \begin{cases} 1 \text{ if } x \geq 0, \\ 0 \text{ if } x < 0, \end{cases}$$

where

$$f(u) = \sum_{k=1}^{\infty} p_k u^k - u,$$

and

$$\sum_{k=1}^{\infty} p_k = 1, \sum_{k=1}^{\infty} k p_k = 2, \text{with } p_k \geq 0.$$

Proof: If we stop the process at $T \wedge t$, where T is the time at which the initial particle splits, then we obtain the following decomposition for $u(t,x)$:

$$
\begin{aligned}
u(t,x) = \mathbb{P}[T > t] \cdot \mathbb{P}[X(t) \le x] + \int_0^t \mathbb{P}[T \in ds] \int_{-\infty}^{\infty} \mathbb{P}[X(s) \in x - dy] \\
\cdot \sum_{k=1}^{\infty} p_k u^k(t-s,y) \\
= e^{-t}\mathbb{P}[X(t) \le x] + \sum_{k=1}^{\infty} p_k \int_0^t e^{-s} ds \int_{-\infty}^{\infty} \mathbb{P}[X(s) \in x - dy]\, u^k(t-s,y) \\
= e^{-t}\mathbb{P}[X(t) \le x] \\
+ \sum_{k=1}^{\infty} p_k \int_0^t e^{-(t-s)} ds \int_{-\infty}^{\infty} \mathbb{P}[X(t-s) \in x - dy]\, u^k(s,y).
\end{aligned}
$$

Substituting s for t-s in the last step, this equation shows that u(t,x) is jointly continuous. Differentiate it with respect to t, and an interchange of limits yields

$$
\begin{aligned}
&\frac{\partial u(t,x)}{\partial t} \\
&= -e^{-t}\mathbb{P}[X(t) \le x] + e^{-t}\frac{\partial \mathbb{P}[X(t) \le x]}{\partial t} + \sum_{k=1}^{\infty} p_k u^k(t,x) \\
&- \sum_{k=1}^{\infty} p_k \int_0^t e^{-(t-s)} ds \int_{-\infty}^{\infty} \mathbb{P}[X(t-s) \in x - dy]\, u^k(s,y) \\
&+ \sum_{k=1}^{\infty} p_k \int_0^t e^{-(t-s)} ds \int_{-\infty}^{\infty} \frac{\partial \mathbb{P}[X(t-s) \in x - dy]}{\partial t} u^k(s,y) \\
&= -e^{-t}\mathbb{P}[X(t) \le x] + \frac{1}{2} e^{-t}\frac{\partial^2 \mathbb{P}[X(t) \le x]}{\partial x^2} + \sum_{k=1}^{\infty} p_k u^k(t,x) \\
&- \sum_{k=1}^{\infty} p_k \int_0^t e^{-(t-s)} ds \int_{-\infty}^{\infty} \mathbb{P}[X(t-s) \in x - dy]\, u^k(s,y) \\
&+ \frac{1}{2}\sum_{k=1}^{\infty} p_k \int_0^t e^{-(t-s)} ds \int_{-\infty}^{\infty} \frac{\partial^2 \mathbb{P}[X(t-s) \in x - dy]}{\partial x^2} u^k(s,y) \\
&= -\left(e^{-t}\mathbb{P}[X(t) \le x] + \sum_{k=1}^{\infty} p_k \int_0^t e^{-(t-s)} ds \int_{-\infty}^{\infty} \mathbb{P}[X(t-s) \in x - dy]\, u^k(s,y) \right) \\
&+ \sum_{k=1}^{\infty} p_k u^k(t,x) \\
&+ \frac{1}{2}\frac{\partial^2 \left(e^{-t}\mathbb{P}[X(t) \le x] + \sum_{k=1}^{\infty} p_k \int_0^t e^{-(t-s)} ds \int_{-\infty}^{\infty} \mathbb{P}[X(t-s) \in x - dy]\, u^k(s,y) \right)}{\partial x^2} \\
&= \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2} + \sum_{k=1}^{\infty} p_k u^k(t,x) - u(t,x).
\end{aligned}
$$

Here the second equation is because $\frac{\partial}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}$ is the infinitesimal generator for Brownian motion.

Kolmogorov-Petrovsky-Piscounov proved the first order term of the Maximal Displacement [11].

Theorem 2.2 (Kolmogorov, Petrovsky, and Piscounov'37) [11]: For fixed

$\varepsilon \in (0,1)$,

$$m_\varepsilon(t) = \sqrt{2}t + o(t), \text{as } t \to \infty.$$

Proof: To compute $u(t,x)$ directly,

$$1 - u(t,x) = \mathbb{P}\left[\max_k X_k(t) > x\right] \leq \mathbb{E}[\#X_k(t) > x] = e^t \mathbb{P}[X_k(t) > x]$$

$$= e^t \int_{-\infty}^{x} \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}} dy,$$

we can estimate that

$$1 - u\left(t, x + \sqrt{2}t\right) = \left(1 + o(1)\right) \frac{e^{-\frac{x}{\sqrt{2}}}}{2\sqrt{\pi}}.$$

Therefore,

$$m_\varepsilon(t) = \sqrt{2}t + o(t), \text{as } t \to \infty.$$

2.2 Tightness of the maximal displacement

In this section, Proposition 2.4 shows that $u(t,x)$ is tight when properly centered. And the main tool will be Lemma 2.3.

Lemma 2.3 [13, page 326-327]: For $0 < \varepsilon < 1$, let $m_\varepsilon(t)$ be chosen to satisfy $u\left(t,\ m_\varepsilon(t)\right) = \varepsilon$, then

$$u\left(t, x + \ m_\varepsilon(t)\right) \text{ is } \begin{cases} \text{increasing in } t \text{ if } x < 0, \\ \text{decreasing in } t \text{ if } x > 0. \end{cases}$$

Proof: Fix $t_0 > 0$ and $a > 0$, and let $v(t,x) = u(t+a, x+b) - u(t,x)$ with $b = m_\varepsilon(t_0 + a) - m_\varepsilon(t_0)$. Then

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + kv$$

with

$$k = u(t+a, x+b) + u(t,x) - 1.$$

By (2.1),

$$v(0+,x)\begin{cases}>0 \text{ if } x<0,\\ <0 \text{ if } x>0,\end{cases}$$

and $v(t_0,x_0)=0$ for $x_0=m_\varepsilon(t_0)$. It is to be proved that $v(t_0,x)\le 0$ for $x>x_0$. Then $v'(t_0,x_0)\le 0$, and the lemma will follow from that. This can be proved by contradiction. Suppose $v(t_0,x_1)>0$ for some $x_1>x_0$, then (t_0,x_1) must be connected to $(t=0)\times(x<0)$ by a continuous curve C along which $v>0$, by means of Kac's formula:

$$v(t_0,x)=\mathbb{E}\left[\exp\left\{\int_t^{t_0}k(t_0-t,X(t))\,dt\right\}v\big(t_0-t,X(t)\big)\right],$$

where X is a standard Brownian motion starting at $t(0)=x$.

Fix $x=x_1$, and looking backwards from t_0, the first root $t\le t_0$ defines a stopping time, then the right-hand side vanishes, contradicting $v(t_0,x_1)>0$.

Fix such a curve C and use the formula with $x=x_0$ and t the passage time to C, then the expectation is positive while the left-hand side vanished.

So the only way to avoid the contradiction is to admit that $v(t_0,x_1)>0$ cannot be maintained. Hence the proof is finished.

Proposition 2.4 [5, page 571-574]: The following convergence holds uniformly in x

$$u\big(t,x+\ m_\varepsilon(t)\big)\to w_\varepsilon(x) \text{ as } t\to\infty,$$

where $w_\varepsilon(x)$ is the unique distribution function which solves the ordinary differential equation

$$0=\frac{1}{2}w_{xx}+\sqrt{2}w_x+f(w).$$

Proof: Lemma 2.3 implies that $\lim_{t\to\infty}u\big(t,x+\ m_\varepsilon(t)\big)=w_\varepsilon(x)$ exists. According to the decomposition for $u(t,x)$ in Theorem 2.1, $u(t,x)$ is continuous in x for $t>0$. Therefore, $w_\varepsilon(x)$ is continuous, and hence convergence is uniform in x.

Next we will show that that $w_\varepsilon(x)$ is the unique distribution function by contradiction that $w_\varepsilon(-\infty)=0$.

The proof that $w_\varepsilon(\infty)=1$ is analogous. Actually, if $w_\varepsilon(\infty)=\alpha<1$, then

$w_\beta(-\infty) > 0$ where $\beta = \frac{1}{2}(1+\alpha)$.

The basic idea is to show that $\frac{m_\varepsilon(t)}{t} \to \infty$ as $t \to \infty$, which contradicts Theorem 2.2.

First stop the process at $T \wedge 1$, and we obtain

$$u(t,x) = e^{-1}\int_{-\infty}^{\infty} \mathbb{P}[x(1) \in -dy]\, u(t-1, x+y) + \int_0^1 ds\, e^{-s} \int_{-\infty}^{\infty} \mathbb{P}[x(s) \in -dy]$$

$$\cdot \sum_{k=1}^{\infty} p_k u^k(t-s, x+y), \qquad \text{for } t \geq 1.$$

Choosing proper M and δ, after some calculation we have that

$$\int_{-\infty}^{\infty} \mathbb{P}[X(1) \in -dy]\, u\left(t-1, m_\beta(t-1) + \frac{1}{2}M + y\right) \leq \beta + 2\delta,$$

and

$$\int_{-\infty}^{\infty} \mathbb{P}[X(s) \in -dy] \sum_{k=1}^{\infty} u^k \left(t-s, m_\beta(t-s) + \frac{1}{2}M + y\right) \leq \beta - 2\delta, \text{for } 0 \leq s \leq 1.$$

If we define

$$\widehat{m}_\beta(t) = \inf_{0 \leq s \leq 1} m_\beta(t-s),$$

then it can be shown that

$$\widehat{m}_\beta(t) + \frac{1}{2}M < \widehat{m}_\beta(t+1).$$

Repeat the above result, we obtain

$$\lim_{t\to\infty} \frac{\widehat{m}_\beta(t)}{t} > \frac{1}{2}M.$$

Let $M \to \infty$, we conclude that

$$\lim_{t\to\infty} \frac{\widehat{m}_\beta(t)}{t} = \infty.$$

So the proof is complete.

2.3 The second order term of the maximal displacement

Theorem 2.5 (Bramson'77) [5, page 574-576]: For fixed $\varepsilon \in (0,1)$,

$$m_\varepsilon(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + b_\varepsilon(t),$$

where $b_\varepsilon(t) = O(1)$ as $t \to \infty$.

Bramson's proof of Theorem 2.5 is based on the following two propositions: the upper and lower bound for the maximal displacement.

Proposition 2.6 [5, page 556]: For $0 \le y \le \sqrt{t}$ and $t \ge 2$,

$$1 - u\left(t, \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + y\right) \le \gamma(y+1)^2 e^{-\sqrt{2}y},$$

where γ is independent of t and y.

We can choose $\gamma' \ge 1$ large enough such that

$$\gamma(y+1)^2 e^{-\sqrt{2}y} \le \gamma' e^{-y}$$

for all $y \ge 0$.

Proposition 2.7 [5, page 568]: For $0 \le y \le \sqrt{t}$ and $t \ge 2$,

$$1 - u\left(t, \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + y\right) \ge \delta e^{-\sqrt{2}y},$$

where $\delta > 0$ is independent of t and y.

These two propositions together imply that

$$\text{(2.2)} \qquad \delta e^{-\sqrt{2}y} \le 1 - u(t, \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + y) \le \gamma' e^{-y}$$

for $0 \le y \le \sqrt{t}$ and $t \ge 2$.

Recall that $u\big(t, m_\varepsilon(t)\big) = \varepsilon$, and let $y = m_\varepsilon(t) - (\sqrt{2}t - \frac{3}{2\sqrt{2}}\log t)$. By inverting (2.2), we see that

$$\frac{1}{\sqrt{2}}\log\frac{\delta}{1-\varepsilon} \le m_\varepsilon(t) - (\sqrt{2}t - \frac{3}{2\sqrt{2}}\log t) \le \log\frac{\gamma'}{1-\varepsilon}$$

for $1 - \delta < \varepsilon < 1 - \gamma' \exp(-\sqrt{t})$. As $t \to \infty$, the conclusion holds for $1 - \delta < \varepsilon < 1$.

Note that if $0 < \varepsilon < \varepsilon_0 < 1$, then $m_\varepsilon(t) \le m_{\varepsilon_0}(t)$. As $t \to \infty$,

$$\varepsilon_0 = u\left(t, m_{\varepsilon_0}(t)\right) = u(t, m_{\varepsilon_0}(t) - m_\varepsilon(t) + m_\varepsilon(t)) \downarrow w_\varepsilon(m_{\varepsilon_0}(t) - m_\varepsilon(t))$$

so

$$0 \le m_{\varepsilon_0}(t) - m_\varepsilon(t) \le w_\varepsilon^{-1}(\varepsilon_0) < \infty.$$

Therefore, if we choose $\varepsilon_0 > 1 - \delta$, it follows that, for $t \ge \log^2(\frac{\gamma'}{1-\varepsilon_0})$,

$$\frac{1}{\sqrt{2}}\log\frac{\delta}{1-\varepsilon_0} - w_\varepsilon^{-1}(\varepsilon_0) \le m_{\varepsilon_0}(t) - w_\varepsilon^{-1}(\varepsilon_0) - \left(\sqrt{2}t - \frac{3}{2\sqrt{2}}\log t\right)$$
$$\le m_\varepsilon(t) - \left(\sqrt{2}t - \frac{3}{2\sqrt{2}}\log t\right) \le \log\frac{\gamma'}{1-\varepsilon} \le \log\frac{\gamma'}{1-\varepsilon_0},$$

which implies Theorem 2.4 for $0 < \varepsilon \le 1 - \delta$ as well.

To prove Proposition 2.6, firstly the probability space Ω is partitioned into small sets, and conditioned on each set, the expectation of the number of particles above a certain curve is estimated. Then combining these estimations together, the upper bound is obtained.

Proof of Proposition 2.6 [5, page 559-562]: Let $e_i = t - \exp\{\frac{2^{\frac{3}{2}}(i_0 - i)}{3}\}$, $i \in \mathbb{N}$, where i_0 is chosen so that $e_0 \ge \frac{5}{6}t > e_{-1}$.

Set

$$S = \inf\left\{s: 0 \le s \le t, \max_{1\le k\le n(t)} X_k(s) - \left(\sqrt{2}t - \frac{3}{2\sqrt{2}}\frac{s}{t}\log t\right) \ge L(s) + y\right\},$$

where L(s) is defined as

$$L(s) = \begin{cases} 0 \quad \text{if } 0 \le s \le 1 \text{ or } t-1 \le s \le t, \\ \frac{3}{2\sqrt{2}}\log s \qquad\qquad \text{if } 1 \le s \le \frac{t}{2}, \\ \frac{3}{2\sqrt{2}}\log(t-s) \qquad \text{if } \frac{t}{2} \le s \le t. \end{cases}$$

S is a stopping time. Define the sets of events A_i such that

$$A_0 = \{S \le e_0\},$$
$$A_i = \{e_{i-1} < S \le e_i\} \text{ for } i = 1, \dots, i_0,$$

$$A_{i_0+1} = \{t - 1 < S \le t\},$$

$$A_{i_0+2} = \{S = \infty\}.$$

Obviously, $\bigcup_{i=0}^{i_0+2} A_i = \Omega$.

Bramson managed to prove that

$$\mathbb{P}[A_0] \le c_1 e^{-\sqrt{2}y},$$

$$\mathbb{P}[A_i] \le c_2(y+1)^2 e^{-\sqrt{2}y} \exp\{-\frac{2^{\frac{1}{2}}(i_0+1-i)}{3}\},$$

$$\mathbb{P}\left[A_{i_0+1}\right] \le c_3(y+1)^2 e^{-\sqrt{2}y}.$$

Therefore,

$$1 - u\left(t, \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + y\right) \le \mathbb{P}[S < \infty]$$
$$\le \left(c_1 + c_2 \sum_{i=1}^{i_0} \exp\left\{-\frac{2^{\frac{1}{2}} i}{3}\right\} + c_3\right)(y+1)^2 e^{-\sqrt{2}y} \le \gamma(y+1)^2 e^{-\sqrt{2}y},$$

where

$$\gamma = c_1 + c_2 \sum_{i=1}^{\infty} \exp\left\{-\frac{2^{\frac{1}{2}} i}{3}\right\} + c_3.$$

Proof of Proposition 2.7 [5, page 568-570]: Let h be the number of branches $X_k(t)$ whose paths satisfy

$$\sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + y < X_k(t) \le \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + y + 1,$$

and

$$X_k(s) < \frac{s}{t}\left(\sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + y + 1\right) + 1 - L(s) \text{ for } 0 \le s \le t,$$

where $L(s)$ is defined in Proposition 2.6.

Bramson proved in his paper that

$$\mathbb{E}[h] \ge d_1 e^{-\sqrt{2}y},$$

and

$$\mathbb{E}[h^2] \leq d_2 E[h].$$

Therefore,

$$1 - u\left(t, \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + y\right) \geq \mathbb{P}[h > 0] \geq \frac{(\mathbb{E}[h])^2}{\mathbb{E}[h^2]} \geq \frac{\mathbb{E}[h]}{d_2} \geq \frac{d_1}{d_2} e^{-\sqrt{2}y} = \delta e^{-\sqrt{2}y},$$

where

$$\delta = \frac{d_1}{d_2}.$$

Chapter 3 Other works on the maximal displacement of a branching Brownian motion

3.1 Roberts's result on the distribution of M_t

Roberts [15] provided another proof of the following result which is much shorter than Bramson's proof.

Define

$$u(t,x) = \mathbb{P}[M_t \leq x].$$

Recall the result that u converges to a travelling wave: there exist function m of t and w of x such that

$$u(t, m(t) + x) \to w(x)$$

uniformly in x as $t \to \infty$.

Theorem 3.1 [15, page 2]: The centering term $m(t)$ satisfies

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + O(1)$$

as $t \to \infty$.

Same as Bramson, Roberts also gives the lower bound and upper bound for $m(t)$. See the following two propositions.

Proposition 3.2 [15, page 9]: There exist t_0 and a constant $\delta \in (0,\infty)$ not depending on t or y such that

$$\mathbb{P}\left(\exists 1 \leq k \leq N(t) \colon X_k(t) \geq \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + y\right) \geq \delta e^{-\sqrt{2}y}$$

for all $y \in [0, \sqrt{\gamma t}]$ and $t \geq t_0$.

Proposition 3.3 [15, page 10]: There exist t_0 and a constant $A \in (0,\infty)$ not depending on t or y such that

$$\mathbb{P}\left(\exists 1 \leq k \leq N(t): X_k(t) \geq \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + y\right) \leq A(y+2)^4 e^{-\sqrt{2}y}$$

for all $y \in [0,\sqrt{t}]$ and $t \geq t_0$.

Having shown that

$$\delta e^{-\sqrt{2}y} \leq 1 - u\left(t, 2^{\frac{1}{2}}t - 3 \cdot 2^{-\frac{3}{2}}\log t + y\right) \leq A(y+2)^4 e^{-\sqrt{2}y},$$

it can be deduced that $m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + O(1)$.

Bramson proved by estimating $G(t)$ directly, where $G(t)$ is the number of particles near $m(t)$ at time t. In Robert's work, he estimated $H(t)$, the number of particles near $m(t)$ that have remained below $\frac{s}{t}m(t)$ for all times $s < t$.

We know that if $B_t, t \geq 0$ is a Brownian motion in $\mathbb{R}^3$, then its modulus $|B_t|, t \geq 0$ is called a Bessel-3 process. Robert guessed that particles behaving in this way look like Bessel-3 processes below the line $\frac{s}{t}m(t), 0 < s < t$. Thanks to this observation, Roberts obtained the upper and lower bound by estimating on Bessel processes using a change of measure, which is based on the many-to-one lemma and many-to-two lemma. In that, Roberts gives proofs much shorter and simplier than those of Bramson.

Lemma 3.4 (Simple many-to-one lemma) [6, page 1]: For measurable f,

$$\mathbb{E}\left[\sum_{k=1}^{N(t)} f\big(X_k(t)\big)\right] = e^t \mathbb{E}[f(B_t)].$$

Lemma 3.5 (Simple many-to-two lemma) [6, page 2]: For measurable f and g,

$$\mathbb{E}\left[\sum_{i,j=1}^{N(t)} f\big(X_i(t)\big)\, g\left(X_j(t)\right)\right] = e^{2t}\mathbb{E}\left[e^{T\wedge t} f(B_t) g(B'_t)\right],$$

where

$$B'_t = \begin{cases} B_t & \text{if } t < T \\ B_T + W_{t-T} & \text{if } t \geq T \end{cases}$$

with T exponentially distributed with parameter 2, and $W_t, t \geq 0$ a standard Brownian motion independent of B_t.

Lemma 3.6 (General many-to-one lemma) [6, page 7]: Let $f_t(\cdot)$ be a measurable function of t and the path of a particle up to time t. Fix $\alpha > 0$ and $\beta \in \mathbb{R}$ and define $\zeta(t) = \frac{1}{\alpha}(\alpha + \beta t - \xi_t)e^{\beta\xi_t - \frac{1}{2}\beta^2 t}\mathbf{1}_{\{\xi_s < \alpha + \beta s, \forall s \leq t\}}$ for $t \geq 0$, then

$$\mathbb{E}\left[\sum_{k=1}^{N(t)} f_t(X_k(t))\right] = e^t \mathbb{Q}[\frac{1}{\zeta(t)} f_t(\xi_t)]$$

where under $\mathbb{Q}$, $\alpha + \beta t - \xi_t, t \geq 0$ is a Bessel process.

Lemma 3.7 (General many-to-two lemma) [6, page 8]: Let $f_t(\cdot)$ and $g_t(\cdot)$ be two measurable functions of t and the path of a particle up to time t. Fix $\alpha > 0$ and $\beta \in \mathbb{R}$ and define $\zeta_i(t) = \frac{1}{\alpha}\left(\alpha + \beta t - \xi_t^i\right)e^{\beta\xi_t^i - \frac{1}{2}\beta^2 t}\mathbf{1}_{\{\xi_s^i < \alpha + \beta s, \forall s \leq t\}}$ for $i = 1{,}2$ and $t \geq 0$, then

$$\begin{aligned} \mathbb{E}\left[\sum_{i,j=1}^{N(t)} f_t(X_i(t))\, g_t\left(X_j(t)\right)\right] \\ &= e^{2t}\mathbb{Q}\left[\frac{1}{\zeta^1(t)} f_t(\xi_t^1) g_t(\xi_t^1)\right] \\ &+ \int_0^t 2\, e^{2t-s}\mathbb{Q}\left[\frac{\zeta^1(s)}{\zeta^1(t)\zeta^2(t)} f_t(\xi_t^1) g_t(\xi_t^2) | T = s\right] ds \end{aligned}$$

where under $\mathbb{Q}$, $\alpha + \beta t - \xi_t^1$ and $\alpha + \beta t - \xi_t^2$ are Bessel processes started from α; fix a time $T \in [0, \infty)$, $\xi_t^1 = \xi_t^2$ for all $t \leq T$, $(\xi_t^1 - \xi_T^1, t > T)$ and $(\xi_t^2 - \xi_T^2, t > T)$ are independent given T and ξ_T^1.

Proof of Proposition 3.2 [6, page 8-10]: For $t > 0$ set

$$\beta = \sqrt{2} - \frac{3}{2\sqrt{2}}\frac{\log t}{t} + \frac{y}{t}.$$

Define

$$H_\alpha(t) = \#\{1 \leq k \leq N(t) : X_k(t) \leq \alpha + \beta s \ \forall s \leq t, \beta t - 1 \leq X_k(t) \leq \beta t\},$$

and

$$B_i = \{\beta t - 1 \leq \xi_t^i \leq \beta t\} \text{ for } i = 1,2.$$

For large t,

$$\mathbb{E}[H_\alpha(t)] = e^t \mathbb{Q}\left[\frac{1}{\zeta^1(t)} \mathbf{1}_{B_1}\right] \sim e^{t - \frac{1}{2}\beta^2 t} \mathbb{Q}[B_1] \sim t^{\frac{3}{2}} e^{-\sqrt{2}y} \mathbb{Q}[\alpha \leq \alpha + \beta t - \xi_t^1 \leq \alpha + 1]$$
$$\sim \alpha^2 e^{-\sqrt{2}y}.$$

So we can estimate that $\mathbb{E}[H_1(t)] \geq c_1 e^{-\sqrt{2}y}$ for some $c_1 > 0$.

For the second moment of $H_1(t)$,

$$\mathbb{E}[H_1(t)^2] = e^t \mathbb{Q}\left[\frac{1}{\zeta^1(t)} 1_{B_1}\right] + \int_0^t 2\, e^{2t-s} \mathbb{Q}\left[\frac{\zeta^1(s)}{\zeta^1(t)\zeta^2(t)} \mathbf{1}_{B_1 \cap B_2} | T = s\right] ds$$
$$\leq \mathbb{E}[H_1(t)]$$
$$+ 2e^{2t - \beta^2 t + 2\beta} \int_0^t e^{-s} \mathbb{Q}\left[(\beta s + 1 - \xi_s^1) e^{\beta \xi_s^1 - \frac{1}{2}\beta^2 s} \mathbf{1}_{B_1 \cap B_2} \middle| T = s\right] ds.$$

After some calculation we will get

$$\mathbb{E}[H_1(t)^2] \leq c_2 \mathbb{E}[H_1(t)].$$

We deduce that

$$\mathbb{P}[H_1(t) \neq 0] \geq \frac{\mathbb{E}[H_1(t)^2]}{\mathbb{E}[H_1(t)]^2} \geq \delta e^{-\sqrt{2}y}$$

as required.

Proof of Proposition 3.3 [6, page 10-11]: For the upper bound on $m(t)$, the first moment method for $H_\alpha(t)$ is combined with an estimate of the probability that a particle ever moves too far from the origin.

Define

$$\Gamma = \#\{1 \leq k \leq N(t) : X_k(t) \leq \alpha + \beta s + 1 \ \forall s \leq t, \beta t - 1 \leq X_k(t) \leq \beta t + \alpha\}.$$

Note that Γ is very similar to H_α, and we can estimate that

$$\mathbb{E}[\Gamma] \le c_3(\alpha+1)^4 e^{-\sqrt{2}y}$$

for some constant c_3 not depending on t, α or y.

Define also

$$B = \{\exists 1 \le k \le N(t), s \le t: X_k(t) > \beta s + \alpha\}.$$

We claim that for $\alpha \ge y \ge 0$,

$$\mathbb{E}[\Gamma | B] \ge c_4$$

for some constant $c_4 > 0$ also not depending on t, α or y.

Then for $\alpha \ge y \ge 0$,

$$\mathbb{P}[B] \le \frac{\mathbb{E}[\Gamma]\mathbb{P}[B]}{\mathbb{E}[\Gamma 1_B]} = \frac{\mathbb{E}[\Gamma]}{\mathbb{E}[\Gamma|B]} \le \frac{c_3}{c_4}(\alpha+1)^4 e^{-\sqrt{2}y}.$$

Applying Markov's inequality, we have

$$\mathbb{P}\left(\exists 1 \le k \le N(t): X_k(t) \ge \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + y\right) \le \mathbb{P}[\Gamma \ge 1] + \mathbb{P}[B]$$
$$\le A(\alpha+1)^4 e^{-\sqrt{2}y}$$

as required.

3.2 Roberts's result on the paths of M_t

Theorem 3.8 [6, page 2-3]: The maximum M_t satisfies

$$\frac{M_t - \sqrt{2}t}{\log t} \to -\frac{3}{2\sqrt{2}} \text{ in probability,}$$

and

$$\liminf_{t\to\infty} \frac{M_t - \sqrt{2}t}{\log t} = -\frac{3}{2\sqrt{2}} \text{ almost surely,}$$

$$\limsup_{t\to\infty} \frac{M_t - \sqrt{2}t}{\log t} = -\frac{1}{2\sqrt{2}} \text{ almost surely.}$$

This result is the analogue of a result for discrete-time branching random walks given by Hu and Shi [7]. This result says that most of the time the extremal particle stays near m(t), however a particle will travel much further occasionally.

The proof is made up of four parts: the upper and lower bounds in the two statements. Actually only the lower bound in the second statement requires an amount of work; the other three are the simple application of Proposition 3.3 and Borel-Cantelli Lemma. Reader can refer to [6, page 12-16].

The proof of

$$\limsup_{t\to\infty} \frac{M_t - \sqrt{2}t}{\log t} \geq -\frac{1}{2\sqrt{2}} \text{ almost surely}$$

is similar to the proof of Proposition 3.2.

Let

$$\beta_t = \sqrt{2} - \frac{1}{2\sqrt{2}} \frac{\log t}{t}$$

and

$$V(t) = \{1 \leq k \leq N(t): X_k(s) \leq \beta_t s + 1 \ \forall s \leq t, \beta_t t - 1 \leq X_k(t) \leq \beta_t t\}.$$

Define

$$I_n = \int_n^{2n} \mathbf{1}_{\{V(t)\neq\emptyset\}} dt.$$

Hu and Shi noticed that although the probability that a particle has position much bigger than m(t) at a fixed time t very small, the probability that there exists a tims s between n and 2n such that a particle has position much bigger than m(s) at time s is actually quite large.

Applying the many-to-one lemma and many-to-two lemma, we get

$$\mathbb{E}[I_n] = \int_n^{2n} \mathbb{P}[V(t) \neq \emptyset] dt \geq c',$$

and

$$\mathbb{E}[I_n^2] = 2\int_n^{2n} \int_n^t \mathbb{P}[V(s) \neq \emptyset, V(t) \neq \emptyset] ds\, dt \geq c''.$$

So

$$\mathbb{P}[I_n > 0] \geq \mathbb{P}[I_n \geq \frac{\mathbb{E}[I_n]}{2}] \geq \frac{\mathbb{E}[I_n]^2}{4\mathbb{E}[I_n^2]} \geq c > 0.$$

When n is large, at time $2\delta\log n$ there are at least n^δ particles, all of which have position at least $-2\sqrt{2}\delta\log n$. By the above, the probability that none of these has a descendant that goes above $\sqrt{2}s - \frac{1}{2\sqrt{2}}\log s - 2\sqrt{2}\log n$ for any s between $2\delta\log n + n$ and $2\delta\log n + 2n$ is no larger than

$$(1-c)^{n^\delta}.$$

The result follow by the Borel-Cantelli lemma since $\sum_n (1-c)^{n^\delta} < \infty$.

According to Theorem 3.8, we can claim that if there are two branching Brownian motions starting at x(0)=0, their maximal displacements will alternate in the leading position for infinitely many times. In other words, every particle born in a branching Brownian motion has a descendant particle in the "lead" at some future time.

Theorem 3.9: Suppose two independent branching Brownian motions $(X_1^A(t), \cdots, X_{N^A}^A(t))$ and $(X_1^B(t), \cdots, X_{N^B}^B(t))$ start at X(0)=0, then with probability 1 there exist random times $t_n \uparrow \infty$ such that

$$M^A(t_n) < M^B(t_n)$$

for all n, where

$$M^A(t) = \max_{1 \leq i \leq N^A(t)} X_i^A(t)$$

and

$$M^B(t) = \max_{1 \leq i \leq N^B(t)} X_i^B(t).$$

For the proof, reader can refer to [12, page 1057-1058].

Chapter 4 L.-P Arguin, A. Bovier and N. Kistler's results on the extremal process of branching Brownian motion

L.-P Arguin, A. Bovier and N. Kistler [2,3,4] addressed the genealogy of extremal particles. They proved that in the limit of large time t, extremal particles descend with overwhelming probability from ancestors having split either within a distance of order one from time 0, or within a distance of order one from time t. The results suggest that the extremal process of branching Brownian motion in the limit of large times converges weakly to a cluster point process. The limiting process is a randomly shifted Poisson cluster process.

4.1 Derivative martingale

The so-called derivative martingale is denoted by

$$Z(t) \equiv \sum_{k=1}^{N(t)} \left(\sqrt{2}t - X_k(t)\right) e^{\sqrt{2}X_k(t)-2t}.$$

To elaborate, first we need to prove that $Y(t) = e^{-t}\sum_{k=1}^{N(t)} e^{\beta X_k(t)-\frac{\beta^2}{2}t}$ is a positive martingale. Without loss of generality, we assume that $N(t) = 1$. Recall that $e^{\beta X(t)-\frac{\beta^2}{2}t}$ is a martingale, where $X(t)$ is a Brownian motion. For any $s > 0$,

$$\begin{aligned}\mathbb{E}[Y(t+s)|\mathcal{F}_t] &= \mathbb{E}\left[e^{-t-s}\sum_{k=1}^{N(t+s)} e^{\beta X(t+s)-\frac{\beta^2}{2}(t+s)}\middle|\mathcal{F}_t\right] \\ &= e^{-t-s}\mathbb{E}[N(t+s)|\mathcal{F}_t]e^{\beta X(t)-\frac{\beta^2}{2}t} = e^{-t-s}e^{s}e^{\beta X(t)-\frac{\beta^2}{2}t} = e^{-t}e^{\beta X(t)-\frac{\beta^2}{2}t} \\ &= Y(t).\end{aligned}$$

Take derivative w.r.t. β to $Y(t)$, and let $\beta = \sqrt{2}$, we obtain $Z(t)$ by taking the negation so as to make it positive.

Lalley and Sellke [12] proved that $Z(t)$ converges almost surely to a strictly positive random variable Z, and established the integral representation

$$(4.1) \qquad w(x) = \mathbb{E}\left[e^{-CZe^{-\sqrt{2}t}}\right],$$

for some $C > 0$, where $w(x)$ is the unique distribution function which solves the o.d.e.

$$\frac{1}{2}w_{xx} + \sqrt{2}w_x + w^2 - w = 0.$$

4.2 Localization of paths

Arguin et al's approach towards the genealogy of particles at the edge of branching Brownian motion is based on characterizing, up to a certain level of precision, the paths of extremal particles. As a first step towards a characterization, in this section we conclude that such paths cannot fluctuate too wildly in the upward direction.

First introduce some notation. For $\alpha, \gamma > 0$, set

$$f_{t,\gamma}(s) \stackrel{\text{def}}{=} \begin{cases} s^{\gamma} & 0 \le s \le \frac{t}{2} \\ (t-s)^{\gamma} & \frac{t}{2} \le s \le t \end{cases}.$$

The upper envelope up to time t is defined as

$$U_{t,\gamma}(s) \stackrel{\text{def}}{=} \frac{s}{t}m(t) + f_{t,\gamma}(s),$$

and the entropic envelope is defined as

$$E_{t,\alpha}(s) \stackrel{\text{def}}{=} \frac{s}{t}m(t) - f_{t,\alpha}(s).$$

Theorem 4.1 (Upper Envelope) [2, page 6]: Let $0 < \gamma < \frac{1}{2}$, and $y \in \mathbb{R}$ given. For any $\varepsilon > 0$, there exists $r_u = r_u(\gamma, y, \varepsilon)$ such that for $r \ge r_u$ and for any $t > 3r$,

$$\mathbb{P}\left[\exists 1 \le k \le N(t) \colon X_k(s) > y + U_{t,\gamma}(s), \text{for some } s \in [r, t-r]\right] < \varepsilon.$$

This theorem says that the vast majority of particles lies under the upper envelope on the time interval $[r, t-r]$, see Figure 4.1.

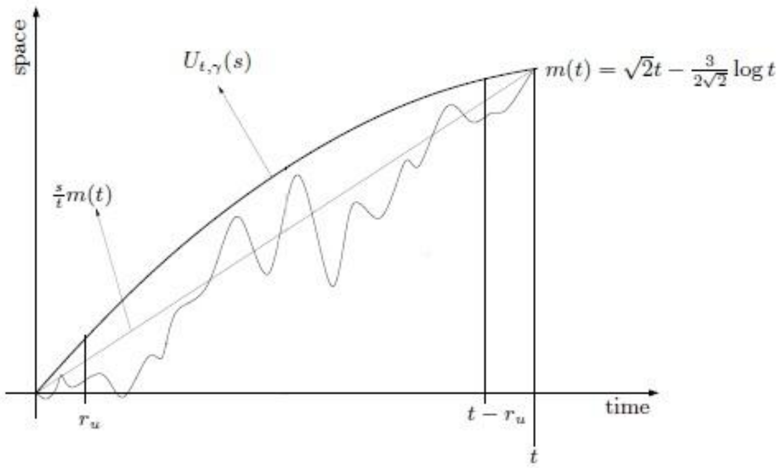

Fig 4.1 The upper envelope [2, page 6]

Theorem 4.2 (Entropic Repulsion) [2, page 8]: Let $0 < \alpha < \frac{1}{2}$, $D \subset \mathbb{R}$ be a compact set. $\bar{D} \stackrel{\text{def}}{=} \sup\{x \in D\}$. For any $\varepsilon > 0$, there exists $r_e = r_e(\alpha, D, \varepsilon)$ such that for $r \geq r_e$ and for any $t > 3r$,

$$\mathbb{P}\left[\exists 1 \leq k \leq N(t): X_k(t) \in m(t) + D, \text{but } \exists s \in [r, t-r]: X_k(s) \geq \bar{D} + E_{t,\alpha}(s)\right] < \varepsilon.$$

Theorem 4.3 (Lower Envelope) [2, page 9]: Let $\frac{1}{2} < \beta < 1$, $D \subset \mathbb{R}$ be a compact set. $\bar{D} \stackrel{\text{def}}{=} \sup\{x \in D\}$. For any $\varepsilon > 0$, there exists $r_l = r_l(\beta, D, \varepsilon)$ such that for $r \geq r_l$ and for any $t > 3r$,

$$\mathbb{P}\left[\exists 1 \leq k \leq N(t): X_k(t) \in m(t) + D, \text{but } \exists s \in [r, t-r]: X_k(s) \leq \bar{D} + E_{t,\beta}(s)\right] < \varepsilon.$$

These two theorems together suggest that the genealogy of particles in $m(t) + \bar{D}$ at time t lies between the entropic envelope and the lower envelope on the time interval $[r, t-r]$, see Figure 4.2.

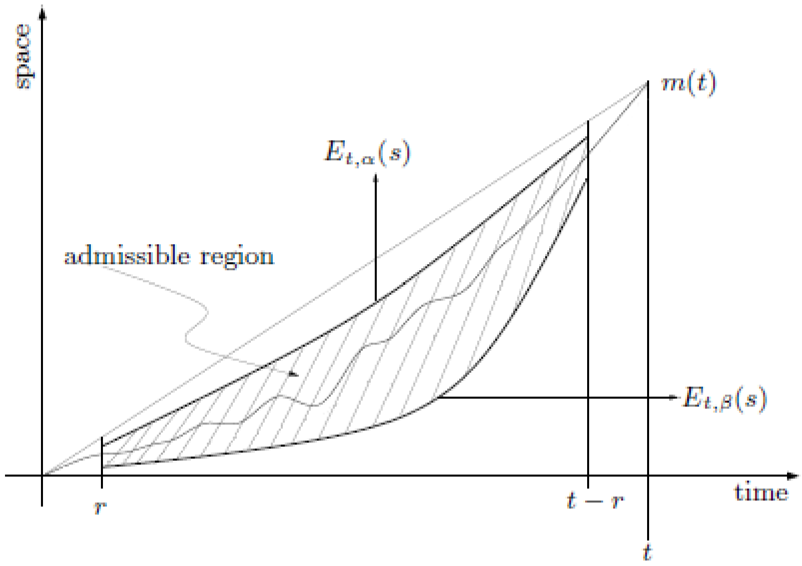

Fig 4.2 The entropic envelope and the lower envelope [2, page 9]

The proof of Theorem 4.1-4.3 is a bit technical, mainly relying on Markov property and the property of the Brownian bridge. The reader is referred to [2, page 17-25].

4.3 The genealogy of extremal particles

Define the genealogical distance

$$Q_{ij}(t) \equiv \inf\{s \le t: X_i(s) \neq X_j(s)\},$$

the time to first branching of the common ancestor.

Theorem 4.4 (The genealogy of extremal particles) [3, page 3]: For any compact $D \subset R$,

$$\lim_{r_d, r_g \to \infty} \sup_{t > 3\max\{r_d, r_g\}} \mathbb{P}[\exists i, j \in \Sigma_t(D): Q_{ij}(t) \in (r_d, t - r_g)] = 0,$$

where $\Sigma_t(D) \overset{\text{def}}{=} \{k \in \Sigma_t: X_k(t) \in m(t) + D\}$, $\Sigma_t = \{1, \cdots, N(t)\}$.

This theorem indicates that extremal particles descend from common ancestors which either branch off "very early" (in the interval $(0, r_d)$), or "very late" (in the interval $(t - r_g, t)$). The next theorems will show that in the middle of the time, the extremal particles should stay within a small area.

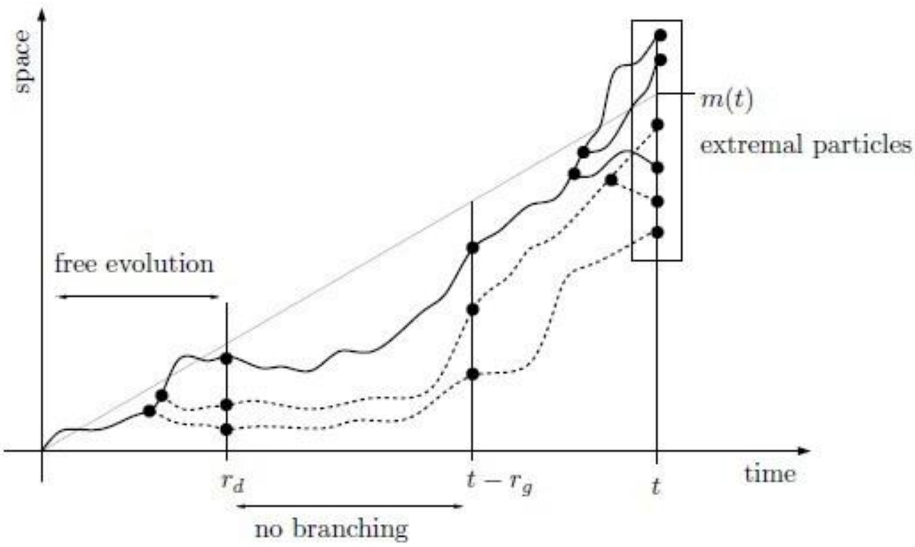

Fig 4.3 Genealogies of extremal particles [3, page 4]

The proof of Theorem 4.4 is based on the localization of the paths.

Proof of Theorem 4.4: Consider the expected number of pairs of particles of branching Brownian motion whose path satisfies some conditions, say $\Xi_{D,t}$,

$$\mathbb{E}\left[\#\{(i,j) i \neq j: X_i, X_j \in \Xi_{D,t}, Q_t(i,j) \in (r_d, t - r_g)\}\right]$$

$$= Ke^t \int_{r_d}^{t-r_g} e^{t-s} \, ds \int_{-\infty}^{\infty} d\mu_s(y) \, \mathbb{P}\left[X \in \Xi_{D,t} \middle| X(s) = y\right] \mathbb{P}\left[X \in \Xi_{D,t}^{s,t-r'} \middle| X(s) = y\right],$$

where μ_s is the Gaussian measure of variance s.

We will show the existence of a $r_0(D, \varepsilon)$ and $t_0(D, \varepsilon)$ such that for $r > r_0$ and $t > \max\{t_0, 3r_d, 3r_g\}$, the right-hand side is small than ε (provided we take $r_0 > r'$).

The idea is to bound the last two terms. Only the basic steps are listed here. The reader is refered to [3, page 8-11].

$$
\mathbb{P}\left[X \in \Xi_{D,t}^{s,t-r'} \middle| X(s) = y\right]
$$

$$
\leq \mathbb{P}\left[\mathfrak{z}_{t-s}(s') \leq \left(1 - \frac{s'}{t-s}\right) Z_1 + \frac{s'}{t-s} Z_2, \forall 0 \leq s' \right.
$$

$$
\left. \leq t - s - r' \right] \mathbb{P}\left[X(t-s) \geq m(t) - y + \underline{D}\right],
$$

$$
\mathbb{P}\left[X \in \Xi_{D,t} \middle| X(s) = y\right] \leq \mathbb{P}\left[\mathfrak{z}_t(s) \leq \overline{D}, \forall r' \leq s \leq t - r'\right] \mathbb{P}\left[X(t) \geq m(t) + \underline{D}\right],
$$

where $\mathfrak{z}_t(s) = X(s) - \frac{s}{t} X(t), 0 \leq s \leq t$ is a Brownian bridge, $\underline{D} \leq \overline{D} \in \mathbb{R}$ are chosen such that $D \subseteq [\underline{D}, \overline{D}]$ for a compact set $D \in \mathbb{R}$.

It can been proved that

$$
\mathbb{P}\left[X \in \Xi_{D,t}^{s,t-r'} \middle| X(s) = y\right] \leq \kappa \sqrt{r} \frac{e^{-(t-s)} e^{\frac{3}{2} \frac{t-s}{t} \log t} f_{t,\beta}(s) e^{-\sqrt{2} f_{t,\alpha}(s)}}{(t-s)^{\frac{3}{2}}},
$$

and

$$
\mathbb{P}\left[X \in \Xi_{D,t} \middle| X(s) = y\right] \leq \kappa r' e^{-t},
$$

for some $\kappa > 0$.

Putting together, we obtain the expectation is less than ε as expected. This implies Theorem 4.4 by Markov's inequality and Theorem 4.2-4.3.

4.4 The thinning map

Define

$$
\overline{Q}(t) = \{\overline{Q}_{ij}(t)\}_{i,j \leq N(t)} \equiv \left\{t^{-1} Q_{ij}(t)\right\}_{i,j \leq N(t)},
$$

$Q_{ij}(t)$ is defined in Section 4.3.

Define also the random measure

$$
\Xi(t) = \sum_{k=1}^{N(t)} \delta_{X_k(t) - m(t)}.
$$

The pair $(\Xi(t), \overline{Q}(t))$ admits the following natural thinning.

For any $q > 0$, the q-thinning of the process $(\Xi(t), \overline{Q}(t))$, denoted by $\Xi^{(q)}(t)$, is defined recursively as follows:

$$\begin{cases} i_1 = 1 \\ \cdots \\ i_k = \min\{j > i_{k-1} : \overline{Q}_{i_l j}(t) < q \ \forall l \le k-1\} \end{cases};$$

and

$$\Xi^{(q)}(t) \equiv \left(\Xi_k^{(q)}(t), k \in \mathbb{N}\right) \equiv \left(\bar{x}_{i_k}(t), k \in \mathbb{N}\right).$$

Then

$$(\Xi(t), \overline{Q}(t)) \longmapsto \Xi^{(q)}(t)$$

is called the thinning map.

To explain, at time $(1-q)t$ every particle X_k will produce a cluster-extrema at time t, and the rule is to pick up the smallest index. Therefore, the cluster-extrema have no common ancestors in the time interval $[(1-q)t, t]$. See figure 4.4.

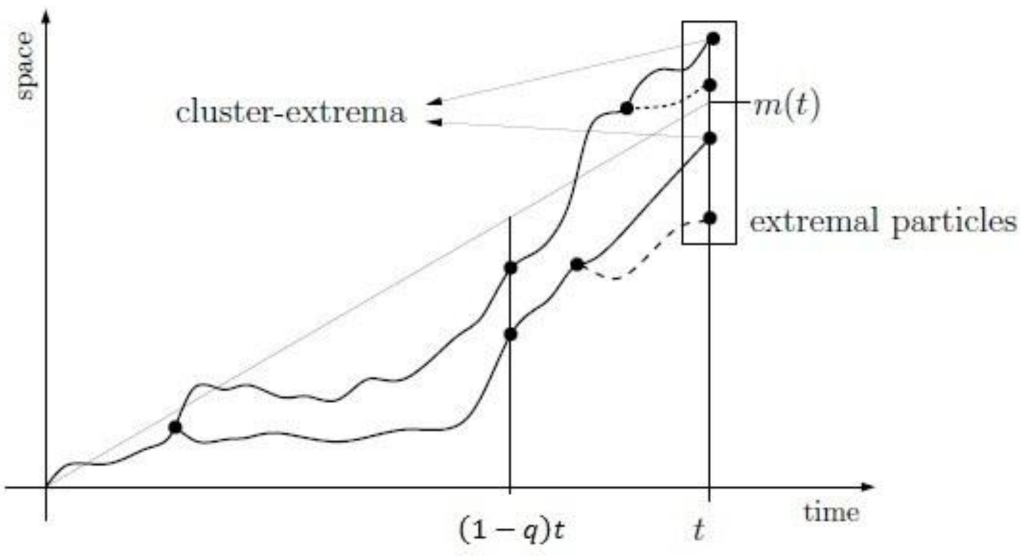

Fig 4.4 Cluster-extrema [3, page 5]

The main result states that all such thinned processes converge to the same randomly shifted Poisson Point Process (PPP) with exponential density.

Theorem 4.5 [3, page 6]: The process $\Xi^{(q)}(t)$ converge in law for any $0 < q < 1$

to the same limit Ξ^0. In particular,

$$\lim_{r_g\to\infty}\lim_{t\to\infty}\Xi^{\left(1-\frac{r_g}{t}\right)}(t)=\Xi^0.$$

Moreover, conditional on Z, the limit of the derivative martingale,

$$\Xi^0=\mathrm{PPP}\left(CZe^{-\sqrt{2}x}dx\right).$$

where $C>0$ is the constant appearing in (4.1).

Proof: Let $\phi:\mathbb{R}\to\mathbb{R}_+$ be measurable, with compact support. We need to show that

$$\lim_{t\to\infty}\mathbb{E}[\exp\{-\int\phi(x)\,\Xi^{(q)}(t)dx\}]=\mathbb{E}\left[\exp\left\{-CZ\int\left(1-e^{-\phi(x)}\right)e^{-\sqrt{2}x}dx\right\}\right],$$

for any $\frac{r_d}{t}\le q\le 1-\frac{r_g}{t}$. This can be obtained by means of the following Lemma 4.6 and Proposition 4.7.

Lemma 4.6 [3, page 7]: For any $y\in\mathbb{R}$ and $\varepsilon>0$, there exists $r_0(D,\varepsilon)$ such that for $r_d,r_g>r_0$ and $t>3\max\{r_d,r_g\}$, on a set of probability $1-\varepsilon$,

$$\Xi^{(q)}(t)|_{(y,\infty)}=\Xi_t^{\left(\frac{r_d}{t}\right)}|_{(y,\infty)},$$

for any $\frac{r_d}{t}\le q\le 1-\frac{r_g}{t}$.

Proposition 4.7 [3, page 8]: With $C>0$ and Z the limiting derivative martingale, conditionally on Z,

$$\lim_{r_g\to\infty}\lim_{t\to\infty}\Xi^{\left(\frac{r_d}{t}\right)}(t)=\mathrm{PPP}\left(CZe^{-\sqrt{2}x}dx\right).$$

To conclude, branching happens at the very beginning, after which particles continue along independent paths, and start branching again only towards the end. The branching at the beginning is responsible for the appearance of the derivative martingale in the large time limit, while the branching towards the end creates the clusters.

4.5 The limit process

The limiting point process of BBM is constructed as follows. Let Z be the limiting derivative martingale. Conditionally on Z, consider the Poisson point process (PPP) of density $CZ\sqrt{2}e^{-\sqrt{2}x}dx$:

$$P_Z \equiv \sum_{i\in\mathbb{N}} \delta_{p_i} \equiv PPP\left(CZ\sqrt{2}e^{-\sqrt{2}x}dx\right).$$

Let $\{X_k(t)\}_{1\le k\le N(t)}$ be a branching Brownian motion of length t. Consider the point process of the gaps $\sum_k \delta_{X_k(t)-M_t}$ conditioned on the event $\{M_t - \sqrt{2}t > 0\}$. Write $\mathcal{D} = \sum_j \delta_{\Delta_j}$ for a point process with this law and consider independent, identically distributed copies $(\mathcal{D}^{(i)})_{i\in\mathbb{N}}$. The following result starts that the extremal process of branching Brownian motion as a point process converges as follows:

Theorem 4.8 [4, page 5]: The family of point processes defined in Section 4.4 converges in distribution to a point process, Ξ, given by

$$\Xi \equiv \lim_{t\to\infty} \Xi_t = \sum_{i.j} \delta_{p_i+\Delta_j^{(i)}}.$$

The key ingredient in the proof of Theorem 4.8 is an identification of the extremal process of branching Brownian motion with an auxiliary process constructed from a Poisson process, with an explicit density of points in the tail.

Next introduce Theorem 4.9 which plays an important role in proving Theorem 4.8.

The auxiliary point process of interest is the superposition of the iid BBM's with drift and shifted by $\frac{1}{\sqrt{2}}\log Z + \eta_i$:

$$\Pi_t \equiv \sum_{i,k} \delta_{\frac{1}{\sqrt{2}}\log Z+\eta_i+X_k^{(i)}(t)-\sqrt{2}t}.$$

Theorem 4.9 (The auxiliary point process) [4, page 6-7]: Let Ξ_t be the extremal process of BBM. Then

$$\lim_{t\to\infty} \Xi_t = \lim_{t\to\infty} \Pi_t.$$

Proof of Theorem 4.8 [4, page 28-29]: The Laplace transform of the extremal process of branching Brownian motion is defined as

$$\psi_t(\phi) \equiv \mathbb{E}\left[\exp\left\{-\int \phi(y)\Xi_t(dy)\right\}\right],$$

for $\phi \in \mathcal{C}_C(\mathbb{R})$ nonnegative.

It suffices to show that for $\phi: \mathbb{R} \to \mathbb{R}_+$ continuous with compact support the Laplace functional of the extremal process of branching Brownian motion satisfies

$$\lim_{t\to\infty} \psi_t(\phi) = \mathbb{E}\left[\exp\left\{-CZ\int_{\mathbb{R}} \mathbb{E}\left[1 - e^{-\int \phi(y+z)\mathcal{D}(dz)}\right]\sqrt{2}e^{-\sqrt{2}y}dy\right\}\right].$$

By Theorem 4.9,

$$\lim_{t\to\infty} \psi_t(\phi) = \lim_{t\to\infty} \mathbb{E}[\exp\{-\sum_{i,k} \phi(\frac{1}{\sqrt{2}}\log Z + \eta_i + X_k^{(i)}(t) - \sqrt{2}t)\}]$$

$$= \mathbb{E}\left[\exp\left\{-Z\lim_{t\to\infty}\int_{-\infty}^{0} \mathbb{E}\left[1 - \exp\left\{-\int \phi(y+z)\overline{\Xi}_t(dx)\right\}\right]\sqrt{\frac{2}{\pi}}\left(-ye^{-\sqrt{2}y}\right)dy\right\}\right],$$

where $\overline{\Xi}_t \equiv \sum_k \delta_{X_k(t)-\sqrt{2}t}$ defined for convenience.

Arguin et al [8, Page 28-29] show that the right-hand side converges to

$$\mathbb{E}\left[\exp\left\{-CZ\int_{\mathbb{R}} \mathbb{E}\left[1 - e^{-\int \phi(y+z)\mathcal{D}(dz)}\right]\sqrt{2}e^{-\sqrt{2}y}dy\right\}\right],$$

which proves Theorem 4.8.

Chapter 5 The work of E.Aidekon, J. Berestycki, E. Brunet and Z. Shi on the branching Brownian motion seen from its tip

The work of E.Aidekon, J. Berestycki, E. Brunet and Z. Shi [1] also gives a description of the limit object and a different proof of the convergence of the branching Brownian motion seen from its tip.

Instead of the previous works where authors usually assume a Brownian motion with variance 1 and no drift, Aidekon et al. assume in their work a Brownian motion with drift 2 and variance 2, while the exponential time with parameter 1 remaining the same.

First, a derivative martingale is also defined. Define

$$Z(t) \equiv \sum_{k=1}^{N(t)} X_i(t) e^{-X_i(t)}.$$

Note that $\mathbb{E}[Z(t)] = 0$.

It can be shown that

$$Z \coloneqq \lim_{t\to\infty} Z(t)$$

exists, is finite and strictly positive with probability 1.

Consider the point process of the particles seen from the rightmost:

$$\bar{\mathcal{N}}(t) \coloneqq \mathcal{N}(t) - m(t) - \log(CZ) = \{X_i(t) - m(t) - \log(CZ), 1 \le i \le N(t)\},$$

where C is the constant appearing in (4.1).

Theorem 5.1 [1, page 4]: As $t \to \infty$ the pair $\{\bar{\mathcal{N}}(t), Z(t)\}$ converges jointly in distribution to $\{\mathcal{L}, Z\}$ where $\mathcal{L}$ and Z are independent and $\mathcal{L}$ is obtained as follows:

(i) Define $\mathcal{P}$ a Poisson point process on $\mathbb{R}$, with intensity measure $e^x dx$;

(ii) For each atom x of $\mathcal{P}$, attach a point process $x+\mathcal{J}^{(x)}$ where $\mathcal{J}^{(x)}$ are independent copies of a certain point process $\mathcal{J}$;

(iii) $\mathcal{L}$ is then the superposition of all the point processes $x+\mathcal{J}^{(x)}$, i.e., $\mathcal{L} \coloneqq \{x+y: x \in \mathcal{P}, y \in \mathcal{J}^{(x)}\}$.

For each $s \leq t$, let $X_{1,t}(s)$ be the position at time s of the ancestor of $X_1(t)$, i.e.,$s \mapsto X_{1,t}(s)$ is the path followed by the rightmost particle at time t. And define

$$Y_t(s) \coloneqq X_{1,t}(t-s) - X_1(t), s \in [0,t]$$

the time reversed path back from the final position $X_1(t)$.

For each $t > 0$ and for each path $X \coloneqq X(s), s \in [0,t]$ that goes from the ancestor to one particle in $\mathcal{N}(t)$, let us write τ_i^X be the successive splitting times of branching along X (enumerated backward), $\mathcal{N}_{t,X}^{(i)}$ be the set of all particles at time t which are descended from the one particle which has branched off X at time τ_i^X relative to the final position $X(t)$, and $\tau_{X,j}(t)$ be the time at which the particle $X_j(t)$ has branched off the path of X during $[0,t]$, then we have

$$\mathcal{N}_{t,X}^{(i)} \coloneqq \left\{X_j(t) - X(t), \tau_{X,j}(t) = \tau_i^X\right\}.$$

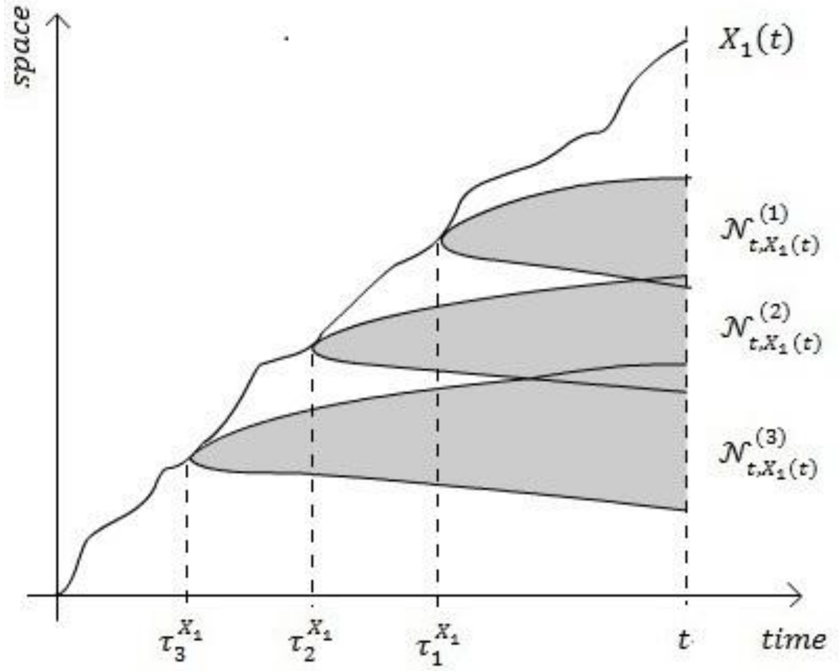

Fig 5.1 The points branching off from X_1 recently

Then define

$$\mathcal{J}(t,\zeta) \coloneqq \bigcup_{\tau_i^{X_1(t)} > t-\zeta} \mathcal{N}^{(i)}_{t,X_1(t)}$$

the set of particles at time t which have branched off $X_{1,t}(s)$ after time $t-\zeta$.

The key result in proving Theorem 5.1 is the following theorem.

Theorem 5.2 [1, page 7-8]: The following convergence holds jointly in distribution.

$$\lim_{\zeta\to\infty}\lim_{t\to\infty}\{(Y_t(s), s\in[0,t]); \mathcal{J}(t,\zeta); X_1(t)-m(t)\} = \{(Y(s), s\geq 0); \mathcal{J}; W\},$$

where the random variable W is independent of the pair $((Y(s), s\geq 0); \mathcal{J})$.

Here, $(Y;\mathcal{J})$ is the limit of the path $s \mapsto X_{1,t}(t-s) - X_1(t)$ and of the points that have branched recently off from $X_{1,t}$. The construction is similar to the thinning map in Chapter 4. Reader can compare Theorem 5.3 to Theorem 4.5.

Next an explicit construction of the decoration point process $\mathcal{J}$ will be given.

Let $B \coloneqq (B_t, t\geq 0)$ be a standard Brownian motion and $R \coloneqq (R_t, t\geq 0)$ be a three-dimensional Bessel process started from $R_0 = 0$ and independent of B. Define $T_b := \inf\{t \geq 0 : B_t = b\}$. For $b > 0$, the process $\Gamma^{(b)}$ is defined as follows:

$$\Gamma_s^{(b)} := \begin{cases} B_s & \text{if } 0 \leq s \leq T_b \\ b - R_{s-T_b} & \text{if } s \geq T_b \end{cases}.$$

The law of the backward path Y is described as follows:

$$\mathbb{P}[Y \in A] = \frac{1}{f(b)} \mathbb{E}\left[e^{-2\int_0^\infty G_v\left(\sigma\Gamma_v^{(b)}\right)dv} \mathbf{1}_{-\sigma\Gamma^{(b)} \in A}\right],$$

where $b \in (0, \infty)$ is a random variable whose density is given by

$$\mathbb{P}[\sigma b \in dx] = \frac{f(x)}{c_1} dx$$

where

$$f(x) = \mathbb{E}\left[e^{-2\int_0^\infty G_v\left(\sigma\Gamma_v^{(x)}\right)dv}\right]$$

and

$$c_1 = \int_0^\infty \mathbb{E}\left[e^{-2\int_0^\infty G_v\left(\sigma\Gamma_v^{(a)}\right)dv}\right] da.$$

Now conditionally on the path Y, let π be a Poisson point process on $[0, \infty)$ with density $2\left(1 - G_t\left(-Y(t)\right)\right)dt = 2(1 - \mathbb{P}_{Y(t)}(X_1(t) < 0))dt$, where $G_t(x) := \mathbb{P}_0[X_1(t) \leq x] = \mathbb{P}_x[X_1(t) \leq 0]$. For each point $t \in \pi$ start an independent branching Brownian motion $(X_{Y(t)}(u), u \geq 0)$ at position $Y(t)$ conditioned to $\min X(t) > 0$. Then $\mathcal{J}$ is defined as

$$\mathcal{J} := \bigcup_{t \in \pi} X_{Y(t)}(t).$$

Now a good event $A_t(x, \eta)$ is defined for it happens with high probability. Fix a constant $\eta > 0$. For $t \geq 1$ and $x > 0$,

$$A_t(x, \eta) := E_1(x, \eta) \cap E_2(x, \eta) \cap E_3(x, \eta)$$

where

$$E_1(x, \eta) := \left\{\forall k \text{ s.t. } |X_k(t) - m(t) < \eta|, \min_{s \in [0,t]} X_{k,t}(s) \geq -x, \min_{s \in \left[\frac{t}{2}, t\right]} X_{k,t}(s) \geq m(t) - x\right\},$$

$$E_2(x,\eta) := \left\{\forall k \text{ s.t. } |X_k(t) - m(t) < \eta|, \forall s \in \left[x, \frac{t}{2}\right], X_{k,t}(s) \geq s^{\frac{1}{3}}\right\},$$

$$E_3(x,\eta) := \left\{\forall k \text{ s.t. } |X_k(t) - m(t) < \eta|, \forall s \in \left[x, \frac{t}{2}\right], X_{k,t}(t-s) - X_k(t) \in [s^{\frac{1}{3}}, s^{\frac{2}{3}}]\right\}.$$

$E_1(x,\eta)$ is the event that all the paths $s \mapsto X_{k,t}(s)$ ending within distance η of $m(t)$ avoid the hashed region on the left. $E_2(x,\eta)$ is the event that those same paths stay above $s^{\frac{1}{3}}$ between x and $\frac{t}{2}$. $E_3(x,\eta)$ is the event that thoses paths stay between $X_k(t) + (t-s)^{\frac{1}{3}}$ and $X_k(t) + (t-s)^{\frac{2}{3}}$ between $\frac{t}{2}$ and $t - x$. See Figure 5.2.

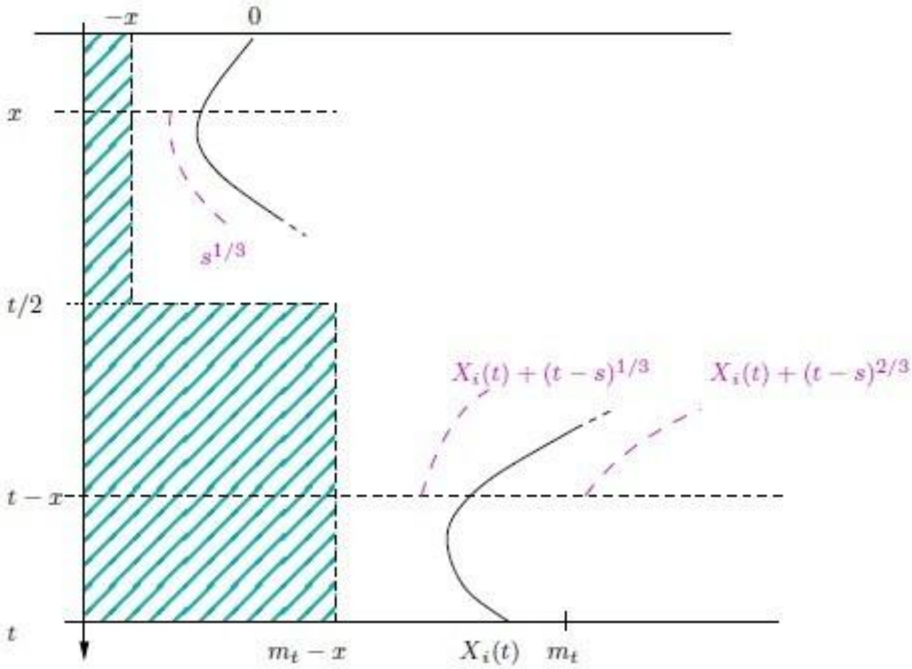

Fig 5.2 The event $A_t(x,\eta)$ [1, page 10]

Theorem 5.3 [1, page 9]: Let $\eta > 0$. For any $\varepsilon > 0$, there exists $x > 0$ large enough such that $\mathbb{P}[A_t(x,\eta)] \geq 1 - \varepsilon$ for t large enough.

Another two propositions are needed in addition. Proposition 5.4 explains the appearance of the point process, and Proposition 5.5 shows that particles sampled near $X_1(t)$ either have a very recent common ancestor or have branched at the very beginning of the process.

Fix $k \geq 1$ and consider $\mathcal{H}_k$ the set of all particles which are the first in their line of descent to hit the spatial position k. Define $H_k := \#\mathcal{H}_k$.

For each $u \in \mathcal{H}_k$, write $X_1^u(t)$ for the maximal position at time t of the particles which are descendants of u and define the finite point process

$$\mathcal{P}_{k,t}^* := (X_1^u(t) - m(t) - \log(CZ_k), u \in \mathcal{H}_k)$$

where

$$Z_k := ke^{-k}H_k.$$

Proposition 5.4 [1, page 11]: The following convergences hold in distribution

$$\lim_{t\to\infty} \mathcal{P}_{k,t}^* = \mathcal{P}_{k,\infty}^* := \sum_{u\in\mathcal{H}_k} \delta_{k+W(u)-\log(CZ_k)}$$

where W(u) are independent copies of the random variable W and

$$\lim_{k\to\infty}\{\mathcal{P}_{k,\infty}^*, Z_k\} = \{\mathcal{P}, Z\}$$

where $\mathcal{P}$ is independent of the random variable Z.

This proposition explains the appearance of the point process $\mathcal{P}$, which can be compared to Theorem 4.9.

Proposition 5.5 [1, page 11-12]: Fix $\eta > 0$ and define an arbitrary deterministic function $k \longmapsto \zeta(k)$ which increases to infinity. We have

$$\lim_{k\to\infty} \lim_{t\to\infty} \mathbb{P}[\mathcal{B}_{\eta,k,t}] = 0.$$

Here the event $\mathcal{B}_{\eta,k,t}$ is defined as

$$\mathcal{B}_{\eta,k,t} := \left\{\exists i, j \in J_\eta(t): \tau_{i,j}(t) \in [\zeta(k), t - \zeta(k)]\right\},$$

where $J_\eta(t) := \{k \in \mathbb{N}: |X_k(t) - m_t| \leq \eta\}$ the set of indices which correspond to particles near m_t at time t, and $\tau_{i,j}(t)$ is the time at which the particles $X_i(t)$ and

$X_j(t)$ have branched from one another (same concept as $Q_{ij}(t)$ in Section 4.3).

This proposition means no branching at intermediate times. Reader can compare it with Theorem 4.4.

Proof of Theorem 5.1 [1, page 10-13]: It can be obtained that

$$Z = \lim_{k\to\infty} Z_k = \lim_{k\to\infty} ke^{-k}H_k$$

almost surely. Let $\bar{\mathcal{N}}_t^{(k)}$ be the extremal process seen from the position $m_t - \log(CZ_k)$

$$\bar{\mathcal{N}}_t^{(k)} \coloneqq \mathcal{N}_t - m(t) - \log(CZ_k).$$

Since

$$\bar{\mathcal{N}}_t^{(k)} \cap (-\eta, \eta) = \bigcup_{u\in\mathcal{H}_k} \left(X_{1,t}^u - m(t) - \log(CZ_k) + \mathcal{J}_{t,\zeta(k)}^{(u)} \cap (-\eta, \eta)\right),$$

and we know that in distribution

$$\lim_{k\to\infty}\lim_{t\to\infty} \bigcup_{u\in\mathcal{H}_k} \left(X_{1,t}^u - m(t) - \log(CZ_k) + \mathcal{J}_{t,\zeta(k)}^{(u)}\right) = \lim_{k\to\infty} \bigcup_{x\in\mathcal{P}_{k,t}^*} \left(x + \mathcal{J}_{\zeta(k)}^{(x)}\right) = \mathcal{J}.$$

where the $\mathcal{J}_{\zeta(k)}^{(x)}$ are independent copies of $\mathcal{J}_{\zeta(k)}$.

Therefore, by Proposition 5.4, we conclude that in distribution

$$\lim_{k\to\infty}\lim_{t\to\infty} \bar{\mathcal{N}}_t^{(k)} \cap (-\eta, \eta) = \mathcal{J} \cap (-\eta, \eta).$$

Hence,

$$\lim_{k\to\infty}\lim_{t\to\infty} \bar{\mathcal{N}}_t^{(k)} = \mathcal{J}$$

in distribution.

Since $\mathcal{N}_t$ is obtained from $\mathcal{N}_t^{(k)}$ by the shift $\log(CZ) - \log(CZ_k)$, which goes to 0, we have in distribution

$$\lim_{t\to\infty} \mathcal{N}_t^{(k)} = \mathcal{J}$$

which leads to the content of Theorem 5.1.

Now let's conclude the results in Chapter 4 and Chapter 5.

	E.Aidekon, J. Berestycki, E. Brunet and Z. Shi's results	L.-P Arguin, A. Bovier and N. Kistler's results
Derivative Martingale	$Z(t) \equiv \sum_{k=1}^{N(t)} X_i(t) e^{-X_i(t)}$	$Z(t) \equiv \sum_{k=1}^{N(t)} \left(\sqrt{2}t - X_k(t)\right) e^{\sqrt{2}X_k(t)-2t}$
Density of PPP	e^x	$CZ\sqrt{2}e^{-\sqrt{2}x}dx$
Auxiliary point process	$\sum_{u\in\mathcal{H}_k} \delta_{k+W(u)-\log(CZ_k)}$	$\sum_{i,k} \delta_{\frac{1}{\sqrt{2}}\log Z+\eta_i+X_k^{(i)}(t)-\sqrt{2}t}$
Extremal process	$\mathcal{N}_t - m(t) - \log(CZ_k)$	$\sum_{k=1}^{N(t)} \delta_{X_k(t)-m(t)}$

Table 5.1 Comparison of results in Chapter 4 and Chapter 5

These results all show that a certain process obtained by a correlation-dependent thinning of the extremal particles converges to a random shift of a Poisson Point Process with exponential density. The difference of the results in two works is mainly because that they assume two kinds of branching Brownian motions in their works. Therefore, although these two works give alternative descriptions of the limit object and proofs of the convergence, they are same in essence.

Bibliography

[1] Aidekon, E., Berestycki, J., Brunet, E., and Shi, Z.. The branching Brownian motion seen from its tip. arXiv: 1104.3738v2 [math. PR], 4 May 2011.

[2] Arguin, L. P., Bovier, A., and Kistler, N.. The Genealogy of Extremal Particles of Branching Brownian Motion. ar Xiv: 1008.4386v2 [math. PR] 27 Jun 2010.

[3] Arguin, L. P., Bovier, A., and Kistler, N.. Poissonian Statistics in the Extremal Process of Branching Brownian Motion. arXiv:1010.2376v2 [math.PR] 17 Nov 2010.

[4] Arguin, L. P., Bovier, A., and Kistler, N.. The Extremal Process of Branching Motion. arXiv: 1103.2322v1 [math. PR] 11 Mar 2011.

[5] Bramson, M. D.. Maximal Displacement of Branching Brownian Motion. Communications on Pure and Applied Mathematics. Volume 31, Issue 5, 531–581, September 1978.

[6] Harris, S. C. and Roberts, M. I.. The many-to-few lemma and multiple spines. arXiv: 1106.4761v2 [math. PR]. 30 Jun 2011.

[7] Hu, Y. and Shi, Z.. Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees. Ann. Probab., 37(2), 742–789, 2009.

[8] Ikeda, N., Nagasawa, M., and Watanabe, S.. Branching Markov processes I. J. Math. Kyoto Univ. Volume 8, Number 2, 233-278, 1968.

[9] Ikeda, N., Nagasawa, M., and Watanabe, S.. Branching Markov processes II. J. Math. Kyoto Univ. Volume 8, Number 3, 365-410, 1968.

[10] Ikeda, N., Nagasawa, M., and Watanabe, S.. Branching Markov Processes III. J. Math. Kyoto Univ. Volume 9, Number 1, 95-160, 1969.

[11] Kolmogorov, A.N., Petrovsky, I.G., and Piskunov, N.S.. Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. Moscou Universitet. Bull. Math. 1, 1-25, 1937.

[12] Lalley, S. P., and Sellke, T.. A conditional limit theorem for the frontier of a branching Brownian motion. The Annals of Probability, Vol. 15, No.3, pp. 1052-1061, 1987.

[13] McKean, H. P.. Application of brownian motion to the equation of kolmogorov-Petrovskii-Piskunov. Communications on Pure and Applied Mathematics. Volume 28, Issue 3, pages 323–331, May 1975

[14] Roberts, M. I.. http://people.bath.ac.uk/mir20/research.html. Matt Roberts' website.

[15] Roberts, M. I.. A simple path to asymptotics for the frontier of a branching Brownian motion. arXiv:1106.4771v2 [math.PR]. 9 Jul 2011.